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AG II - Schemes and sheaves

# Solutions – week 6

**Exercise 1.** *Normal schemes and normalization* An integral scheme $X$ is said to be *normal* if every stalk $\mathcal{O}_{X,x}$ is integrally closed.

(1) Show that an affine integral scheme $\operatorname{Spec}(A)$ is normal if and only if $A$ is normal ring.
(2) Show that an integral scheme is normal if and only for every closed point $x \in U$ the stalk $\mathcal{O}_{X,x}$ is integrally closed for every open affine $U \subset X$.[1]

The *normalization* of an integral scheme $X$ is a scheme $\widetilde{X}$ together with a dominant map[2] $\nu \colon \widetilde{X} \to X$ such that for every dominant morphism from an integral normal scheme $f \colon Z \to X$ there exists a unique morphism $\overline{f} \colon Z \to \widetilde{X}$ with $\nu \overline{f} = f$. Therefore the normalization is unique up to unique isomorphism.

(3) Let $A$ be an integral domain. Show that if $X = \operatorname{Spec}(A)$, then $\operatorname{Spec}(\widetilde{A})$ is the normalization of $X$ if $A \to \widetilde{A}$ is the normalization of $A$.
(4) Show that every integral scheme admits a normalization.

*Solution key.* We first remark the following general fact about integral domains

$$A = \bigcap_{\mathfrak{m} \in \max(A)} A_{\mathfrak{m}}.$$

Indeed, if $x \in \bigcap_{\mathfrak{m} \in \max(A)} A_{\mathfrak{m}}$ the ideal

$$I_x = \{a \in A \mid ax \in A\}$$

needs to contain 1, implying that $x \in A$. Otherwise there is some maximal ideal $\mathfrak{m} \supset I_x$. But as we can write $x = a\lambda^{-1}$ with $a \in A$ and $\lambda \in A \setminus \mathfrak{m}$, we get that $\lambda \in I_x$ a contradiction.

(1) and (2)
Now suppose that for every maximal ideal $\mathfrak{m}$ the local ring $A_{\mathfrak{m}}$ is normal. Write $K = \operatorname{Frac}(A)$. If $a \in K$ is the root of a monic polynomial in $A[t]$, it is therefore also the root of the same monic polynomial seen in $A_{\mathfrak{m}}[t]$, implying that $a \in A_{\mathfrak{m}}$. The above implies that $a \in A$ and as a byproduct, $A$ is normal.
For the converse, we show that any localization of an integral normal ring is again normal. Say $S$ is a multiplicative subset. Take $x \in K$ to be a root of a monic polynomial in $S^{-1}A[t]$. Clearing the denominators and multiplying

[1] For finite type $k$-schemes, this the same as saying every closed point of $X$. See week 10, exercise 1.
[2] A map is called *dominant* if the topological image of the map is dense.

by enough elements of $s$, we see that there is an $s \in S$ such that $sx$ is a root of a monic polynomnial in $A[t]$, implying that $sx \in A$, and that $x \in S^{-1}A$.
(3) Note that first that if $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is dominant with $A$ reduced, it implies that $A \to B$ is injective. Indeed, if $a \mapsto 0$, it implies that $D(a)$ does not meet the image. But then, $D(a) = \emptyset$, implying that $a$ is nilpotent. As $A$ is reduced, the claim follows.
Now the universal property in the category of affine schemes amounts to check equals by duality to the following. If $B$ is normal, and $A \to B$ is injective, then there is a unique factorization $A \to \tilde{A} \to B$. Consider $K_A \to K_B$ the induced map. If $x \in K_A$ is the root of a monic polynomial in $A[t]$, the image in $K_B$ is the root of the same polynomial seen in $B[t]$, implying that the image is in $B$. This concludes.
Now we prove that the universal property also holds in the category of schemes. Let $f\colon Z \to \mathrm{Spec}(A)$ be any dominant map from an integral normal scheme. Cover $Z$ by affine, necessearly normal integral, schemes $(Z_i)$. Then $f_i\colon Z_i \to \mathrm{Spec}(A)$ factors through $\mathrm{Spec}(\widetilde{A})$ by the above. By the universal property, it glues to a necessarily unique map $\widetilde{f}\colon Z \to \mathrm{Spec}(\widetilde{A})$.
(4)

- First we make the important remark for the construction that normalization preserves open immersions. More precisely, if $A \to A'$ is an affine map between integral domains that induces an open immersion, then $\widetilde{A} \to \widetilde{A'}$ also induces an open immersion. The key is that if $S$ is a multiplicative subset of $A$, we have $\widetilde{S^{-1}A} = S^{-1}\widetilde{A}$. Using this we show the claim. That $A \to A'$ is an open immersion means that there exists a finite number of functions $a_1, \dots a_n \in A$ such that the localization $A_{a_i} \to A'_{a_i}$ is an isomorphism, and the image of the $a_i$'s generated the unit ideal in $A'$. Using that we can commute localization and normalization as stated above, we get that the maps $\widetilde{A}_{a_i} \to \widetilde{A}'_{a_i}$ are also isomorphisms also, showing the claim.
- Now we show that any separated integral scheme admits a normalization. Say $X$ is such a scheme, and that $X$ is covered by affine schemes $X_i$'s with affine intersection (by separated) $X_{ij}$. We claim that we can glue the schemes $\widetilde{X}_i$'s to a scheme $\widetilde{X}$ together with a map $\widetilde{X} \to X$. By the above the image of $\varphi_{ij,i}\colon \widetilde{X}_{ij} \to \widetilde{X}_i$ is open. We write it $U_{i,j}$. Now note that $\varphi_{ij,j}\varphi_{ij,i}^{-1}\colon U_{i,j} \to U_{j,i}$ is an isomorphism. We denote this last map $\psi_{i,j}$. Using the universal property of $\widetilde{-}$ in the affine case, it follows that $(\psi_{i,j})_{(i,j)}$ is a collection that satisfies the cocyle condition, allowing us to proceed to the usual gluing construction. Note that the maps $\widetilde{X}_i \to X_i$ glue by construction to a map $\widetilde{X} \to X$. To show that this has the required universal property, let $f\colon Z \to X$ be any dominant map from an integral normal scheme. Write $Z_i = f^{-1}(X_i)$. By the above affine case there is a unique map $\widetilde{f}\colon Z_i \to \widetilde{X}_i \to \widetilde{X}$ which glues to a necessarily unique map $f\colon Z \to \widetilde{X}$ showing the claim.
- The general case follows by the same pattern and the separated case. Namely, any scheme can be covered by an union of open separated (affine) schemes such that the intersection is separated.

□

**Exercise 2.** *Blow-ups.* Let $k$ be an algebraically closed field. You can use the following.
*Let $A = k[x_1, \dots, x_n]/(f)$. Denote by $\partial_i f$ the derivative of $f$ with respect to $x_i$. Then*

$$\operatorname{Spec}(A) \text{ is regular } \iff V(f, \partial_1 f, \cdots, \partial_n f) = \emptyset.$$

*Moreover $V(f, \partial_1 f, \cdots, \partial_n f)$ consists exactly of the non-regular points of* $\operatorname{Spec}(A)$.

(0) Let $R$ be a ring. Show that if $I = (f_0, \dots, f_n)$ is generated by a regular sequence then $\mathrm{Bl}_I = V_+(X_i f_j - X_j f_i)$ in $\mathbb{P}^n_R = \mathbb{P}^n_{\mathbb{Z}} \times \operatorname{Spec}(R)$. (Use the lemmas in the blow-ups document from moodle)
(1) Show that blow-up of $(x^2, y^2)$ in $\operatorname{Spec}(k[x, y])$ is not normal and that the blow-up of $(x, y)$ is its normalization.
(2) Show that blow-up of $(x^2, y)$ in $\operatorname{Spec}(k[x, y])$ is not regular. What are the regular points?[3]
(3) Show that $X = \operatorname{Spec}(k[x, y, z, w]/(xy - zw))$ is not regular. What are the regular points?
(4) Show that blow-ups of $X$ at $(x, y, z, w)$ and $(x, z)$ are regular. We denote these blow-ups by $X_1 \to X$ and $X_2 \to X$.
*Remark.* This is another example where blow-ups resolves (=removes) singularities, as in 4.(3) of week 5.
(5) Compute fibers of $(x, y, z, w)$ of $X_1 \to X$ and $X_2 \to X$.

*Solution key.* This exercise was a previous year hand-in exercise so solutions are credited to past students of the course who wrote them.

**(1)**(Joel) Let $A = k[x, y]$, $I = (x^2, y^2)$ and $R = A/I$. Consider the map $\phi : A[Z, W] \twoheadrightarrow \bigoplus_{n \geq 0} I^n$ which sends $Z \to x^2$ and $W \to y^2$ in degree one. Then $\ker \phi = (Zy - Wx)$, so $\mathrm{Bl}_I \cong \operatorname{Proj}\ A[Z, W]/(Zy^2 - Wx^2)$. Next, we show that the blow-up is not normal. Consider the affine chart $U_W$ where $W \neq 0$, which is given by $\operatorname{Spec} k[x, y, z]/(zy^2 - x^2) =: \operatorname{Spec} B$, where $z = \frac{Z}{W}$. Then $\frac{zy}{x} \in \operatorname{Frac}(B)$, and $\left(\frac{zy}{x}\right)^2 = \frac{z \cdot zy^2}{x^2} = \frac{zx^2}{x^2} = z$. Hence, $\frac{zy}{x}$ is a root of the monic polynomial $P(t) = t - z$ with coefficients in $A$. Now $\frac{zy}{x} \notin B_{((x,y))}$, as $x$ is not inverted in the localization, and the field of fractions of $B_{((x,y))}$ is the same as for $B$, we see that the blow-up is not normal.

The affine chart $U_W$ can also be expressed as $\operatorname{Spec} k[x, y, \frac{x^2}{y^2}]$, and similarly we get a chart $U_Z \cong \operatorname{Spec} k[x, y, \frac{y^2}{x^2}]$. As above, neither of these affine charts are normal, and we can normalize on the ring level by exercise (2). Hence, for $k[x, y, \frac{x^2}{y^2}]$ the normalization is given by $k[x, y, \frac{x^2}{y^2}][\frac{x}{y}] = k[x, y, \frac{x}{y}] = k[y, \frac{x}{y}] = k[y, t]$, and similarly for we get $k[x, t']$ as the normalization for the ring corresponding to $U_Z$. Thus we have two affine planes over $k$ as the normalizations of our charts.

[3]This investigation can be used to show that this blow-up is normal.

Let us inspect the blow-up of $J = (x, y)$: the blow-up algebra $\mathrm{Bl}_J$ is isomorphic to $\tilde{A} := k[x, y][Z, W]/(xW - yZ)$ by the same procedure as in the beginning. Here, we have the charts $\tilde{U}_W = \mathrm{Spec}\ k[x, y, z']/(x - yz') \cong \mathrm{Spec}\ k[x, y, \frac{x}{y}] = \mathrm{Spec}\ k[y, \frac{x}{y}] = k[y, t]$ when $W \neq 0$ and similarly $\tilde{U}_Z = \mathrm{Spec}\ k[x, y, w']/(x - yw') \cong \mathrm{Spec}\ k[x, t']$, which are the normalizations of the two affine charts of the blow up of $(x^2, y^2)$. Now, on the intersection $U_Z \cap U_W$ of $\mathrm{Proj}\ \mathrm{Bl}_I$ we have $Z, W \neq 0$, so $U_Z \cap U_W \cong k[x, y, \frac{x^2}{y^2}, \frac{y^2}{x^2}]$, with its normalization given by $k[x, y, \frac{x}{y}, \frac{y}{x}] = k[y, \frac{x}{y}, \frac{y}{x}]$, which corresponds to the intersection on the blow up of $(x, y)$. Hence, we can glue to get the normalization of the blow-up of $(x^2, y^2)$.

**(2)**(Joel) Let $I = (x^2, y)$ and $R = A/I$. The blow-up is isomorphic to $\mathrm{Bl}_I \cong \mathrm{Proj}\ A[Z, W] = \mathrm{Proj}\ \ k[x, y][Z, W]/(yZ - x^2W)$, which we can cover with $U_Z = \mathrm{Spec}\ k[x, y][a]/(y - ax^2)$ and $U_W = \mathrm{Spec}\ k[x, y][b]/(by - x^2)$, where $a = \frac{W}{Z}$ and $b = \frac{Z}{W}$. On $U_W$ we see that at the point $x = b = 0$ the scheme is not regular by the criterion provided in the exercise, as $(0, 0, 0) \in V(by - x^2, -2x, b, y)$. This is the only non-regular point, as on $U_Z$ we have $V(y - ax^2, -2a, 1, -x^2) = \varnothing$. Hence all points except $x = y = W = 0$ are regular.

**(3)**(Julie) Let $g = xy - zw \in k[x, y, z, w]$. By the criterion provided in the statement of the exercise, the set of non-regular points in

$$\mathrm{Spec}\ \left(\frac{k[x, y, z, w]}{(xy - zw)}\right)$$

is given by

$$V(g, \partial_x g, \partial_y g, \partial_z g, \partial_w g) = V(xy - zw, y, x, -w, -z) = V(x, y, z, w) = \{(x, y, z, w)\},$$

where the last equality holds by maximality of $(x, y, z, w)$ in $k[x, y, z, w]$. Hence, all points of $X$ are regular except for $(x, y, z, w)$ (corresponding to the origin in $V(xy - zw) \subseteq \mathbb{A}^4_k$).

**(4)**(Maxence) Consider $R = k[x, y, z, w]$. Let $I = (x, y, z, w)$, $I' = (x, z)$ and $J = (xy - zw)$. We consider the strict transform $\mathrm{St}_J$ (resp. $\mathrm{St}'_J$) of $V(J) = X$ at $I$ (resp. $I'$) in $\mathbb{A}^4_k$. We denote these schemes as respectively $X_1$ and $X_2$. We know that $X_1$ (resp. $X_2$) is the closed subscheme $V_+(\bigoplus_n I^n \cap J)$ of $\mathrm{Bl}_I$ (resp. the closed subscheme $V_+(\bigoplus_n I'^n \cap J)$ of $\mathrm{Bl}_{I'}$).

Notice that $\mathrm{Bl}_I = \mathrm{Proj}(R[X, Y, Z, W]/\tilde{I})$ and $\mathrm{Bl}_{I'} = \mathrm{Proj}(R[X, Z]/\tilde{I}')$ where

$$\tilde{I} = (yX - xY, zX - xZ, wX - xW, yZ - zY, yW - wY, zW - wZ) \text{ and } \tilde{I}' = (zX - xZ).$$

So, the preimage of the ideal $\bigoplus_n I^n \cap J$ by the natural surjection is given by the ideal $K = \tilde{I} + (xy - zw, xY - zW, XY - ZW)$. Indeed, it must be generated in $R[X, Y, Z, W]$ by homogeneous polynomials with degree less or equal to 2 with respect to the variables $X, Y, Z, W$ whose image is send to the generator of $J$ which has degree 2. These generators are enough since every elements $f$ in $I^n$ has monomials of at least degree $n$ and if $f \in J$, then $f = g \cdot (xy - zw)$. Since $xy - zw$ is of degree 2, the polynomial $g$ must be of degree $n-2$, hence $g \in I^{n-2}$. So for every element in $I^n \cap J$ with $n \geq 3$ can be reach using generators of $K$. In the same way the preimage of $\bigoplus_n I'^n \cap J$ by the natural surjection is the ideal $K' = \tilde{I}' + (xy - zw, yX - wZ)$.

That is,

$$X_1 = \text{Proj}(R[X,Y,Z,W]/K) \text{ and } X_2 = \text{Proj}(R[X,Z]/K').$$

For $X_1$ on $D_+(X)$, we have $\mathcal{O}_{X_1}(D_+(X)) = k[x, s_1, s_2, s_3]/(s_1 - s_2 s_3)$ by simplifying the equations of $K$. And by the criterion, the affine open subset $D_+(X)$ of $X_1$ is regular. The same result holds for $D_+(Y)$,$D_+(Z)$ and $D_+(W)$ by symetry of the variables. Hence $X_1$ is regular.

For $X_2$ on $D_+(X)$, we have $\mathcal{O}_{X_2}(D_+(X)) = k[x, w, s]$ by simplying equations of $K'$, and so $D_+(X) = \mathbb{A}^3_k$ which is regular. The same result holds for $D_+(Z)$ by symetry. Hence $X_2$ is regular.

**(5)**(Maxence) We want to compute the fiber of $f_1 : X_1 \to X$ and $f_2 : X_2 \to X$ over $(x, y, z, w)$.

First, the residue field of $(x, y, z, w) \in X$ is simply $k$ by exactness of localization, so for $i = 1, 2$, we need to compute the fibred product $X_i \times_X \text{Spec}(k)$. Hence, if we denote $A = k[x, y, z, w](xy - zw)$ we have

$$X_1 \times_X \text{Spec}(k) = \text{Proj}\left(\frac{A[X,Y,Z,W]}{K} \otimes_A k\right) \text{ and } X_2 \times_X \text{Spec}(k) = \text{Proj}\left(\frac{A[X,Z]'}{K} \otimes_A k\right).$$

Looking at these tensor products, by using $A$-linearity all relations given by $K$ vanish except $XY - ZW = 0$ in the residue field of $(x, y, z, w)$ by its definition. The same holds for $K'$ but here all its relations vanish.

It yields that

$$X_1 \times_X \text{Spec}(k) = \text{Proj}(k[X,Y,Z,W]/(XY - ZW)) = \mathbb{P}^1_k \times_{\text{Spec}(k)} \mathbb{P}^1_k$$

and

$$X_2 \times_X \text{Spec}(k) = \text{Proj}(k[X,Z]) = \mathbb{P}^1_k.$$

□

**Exercise 3.** *Integrality/reducedness of* Proj. Let $B$ be an $\mathbb{N}$-graded integral/reduced ring. Show that $\text{Proj}(B)$ is an integral/reduced scheme.

*Solution key.* If $B$ is reduced any localization is also. Therefore the degree zero part of any localization by homogeneous elements are also. It implies that $\text{Proj}(B)$ is reduced. If $B$ is integral, the product $ss'$ of two non-zero homegeneous elements $s, s'$ is never zero. It implies that the degree zero part of $B_{ss'}$ is not zero also. It implies that the intersection of two non-empty opens is never empty in Proj($B$). Therefore Proj($B$) is irreducible. Being also reduced, it is integral. □

**Exercise 4.** *Fibers.*

(1) Compute the fibers of the morphism
$$\mathrm{Spec}(\mathbb{Z}[x,y,z]/(2zx+9y^2)) \to \mathrm{Spec}(\mathbb{Z}).$$
Which fiber is reduced ? Which fiber is integral ?

(2) Compute the fibers of the morphism, where $p$ is a prime number
$$\mathrm{Spec}(\mathbb{Z}[x,y]/(xy^2+p)) \to \mathrm{Spec}(\mathbb{Z}).$$
Which fiber is reduced ? Which fiber is integral ?

*Solution key.* (1) The fiber over 2 is not reduced. The fiber over 3 is reduced but not integral. It is integral over any other prime by Eisenstein criterion.

(2) The fiber over $p$ is not reduced and not irreducible. Othewise it is isomorphic to $k[x,y,y^{-1}]$ where $k$ is a prime field not equal to $\mathbb{F}_p$. □

**Exercise 5.** *Properties under base change.* Let $f\colon X \to Y$ be a morphism of schemes. Which of the following properties are stable under base change? Prove the statement or provide a counter-example.

(1) $f$ is an open immersion.
(2) $f$ is a closed immersion.
(3) $f$ is injective.
(4) $f$ has integral fibers.
(5) $f$ has reduced fibers.

*Solution key.* Statements (1) and (2) are true (proof below), for (3) take $\mathrm{Spec}(\mathbb{C}) \to \mathrm{Spec}(\mathbb{R})$ while a counter example to the remaining is the map $\mathrm{Spec}(\mathbb{F}_p(t^{1/p})) \to \mathrm{Spec}(\mathbb{F}_p(t))$, base changed against itself.
Let us start with open immersions. Up to composing by an isomorphism we can suppose that $f\colon X \to Y$ is $U \subset Y$ an open.
But now we see that the following is a pullback diagram
$$\begin{array}{ccc} U' & \longrightarrow & U \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[g]{} & Y \end{array}$$
Let $U' = g^{-1}(U) \subset Y'$ open, equipped with the open-subscheme of $Y'$-structure. Indeed the universal property of the pullback here reads as a map $Z \to Y'$ that topologically factors to the open $f^{-1}(U)$, implying that it factors schematically because the sheaf on the open is just the restriction of the sheaf on the all set.
We now prove and (2). First, a map $f\colon X \to Y$ is a closed immersion if and only if $f\colon f^{-1}(U_i) \to U_i$ is a closed immersion for $\bigcup U_i = Y$ an open cover. Indeed a subset $Z \subset Y$ is closed if and only if $U_i \cap Z \subset U_i$ is closed for every

$i$ and to check that the desired sheaf map is surjective is and only if it is locally.
Therefore if

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \big\downarrow{\scriptstyle f'} & & \big\downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

is a pullback diagram with $f$ being a closed immersion, we can reduce to the affine case as follows. First, take a cover $(U_i)$ of $Y$ by affines, and consider the cover of $X$ induced by the pre-images $(g^{-1}(U_i))$. Then cover each of these opens $g^{-1}(U_i)$ by affines $(V_{ij})$. Then

$$\begin{array}{ccc} f'^{-1}(V_{ij}) & \longrightarrow & f^{-1}(U_i) \\ \big\downarrow{\scriptstyle f'} & & \big\downarrow{\scriptstyle f} \\ V_{ij} & \longrightarrow & U_i \end{array}$$

is again a pullback diagram.
We now use the following lemma.
**Lemma.** *Let* $X = \operatorname{Spec}(A)$ *be affine and* $\iota \colon Z \to X$ *a closed immersion. Then the natural map* $Z \to \operatorname{Spec}(\mathcal{O}_Z(Z))$ *is an isomorphism and*

$$A \to \mathcal{O}_Z(Z)$$

*is surjective. If* $I$ *is the kernel of this map, we therefore have*

$$\begin{array}{ccc} Z & \xrightarrow{\iota} & \operatorname{Spec}(A) \\ \sim\big\downarrow & \nearrow & \\ \operatorname{Spec}(A/I) & & \end{array}$$

*Proof.* Let $Z = \cup_i V_i$ a finite covering by affines. By hypothesis $V_i = U_i \cap Z$ for some open $U_i$ of $X$. Covering all $U_i$ and $X \setminus Z$ by finitely many principal opens of $X$ we can suppose that $V_i = D(f_i) \cap Z$ for some $f_i \in \mathcal{O}_X(X)$ with $(f_i)$ being the unit ideal in $A$, and therefore in $\mathcal{O}_Z(Z)$ also. Now we use week 5.5.2 to conclude that $Z$ is affine. Therefore $Z \to \operatorname{Spec}(\mathcal{O}_Z(Z))$ is an isomorphism.
By assumption for every $\mathfrak{p} \in \operatorname{Spec}(A)$ the map $\mathcal{O}_{X,\mathfrak{p}} \to (\iota_* \mathcal{O}_Z)_{\mathfrak{p}}$ is surjective. When $\mathfrak{p} \notin Z$ the right is zero and coincides with $\mathcal{O}_Z(Z)_{\mathfrak{p}}$: indeed take $\mathfrak{p} \in D(f_i) \subset X \setminus Z$, then as $D(f_i) \cap Z = \emptyset$, we conclude that $f_i$ in $\mathcal{O}_Z(Z)$ is nilpotent and as $\mathcal{O}_Z(Z)_{\mathfrak{p}}$ is a further localization of $\mathcal{O}_Z(Z)_{f_i} = 0$ we have our claim. When $\mathfrak{p} \in Z$ the right hand side is $\mathcal{O}_{Z,\mathfrak{p}}$, and because $X$ and $Z$ are affine this is $A_{\mathfrak{p}} \to \mathcal{O}_Z(Z)_{\mathfrak{p}}$. So we conclude that $A \to \mathcal{O}_Z(Z)$ is a map of $A$-modules surjective at every localization at primes, implying that this map is surjective. $\square$

Therefore $f^{-1}(U_i)$ is also affine. Because the inclusion of affine schemes into schemes preserve limits, we are therefore in a situation

$$\begin{array}{ccc} \mathrm{Spec}(B \otimes_A A/I = B/IB) & \longrightarrow & \mathrm{Spec}(A/I) \\ \downarrow f' & & \downarrow f \\ \mathrm{Spec}(B) & \xrightarrow{\quad g \quad} & \mathrm{Spec}(A) \end{array}$$

which concludes.

□

**Exercise 6.** *An open of an affine is not neccesarly affine.* Let $R$ be a non-zero ring. Show that $U = \mathrm{Spec}(R[x, y]) \setminus V(x, y)$ is not affine.
*Hint: compute $\mathcal{O}(U)$ using an appropriate cover and the sheaf property.*

*Solution key.* We use the cover $D(x) \cup D(y)$ and the sheaf property to compute global sections of $U$. Because $x, y$ are non zero divisors, localization maps $R[x^{\pm 1}, y] \to R[x^{\pm 1}, y^{\pm 1}]$ and $R[x, y^{\pm 1}] \to R[x^{\pm 1}, y^{\pm 1}]$ ar injective and we may treat them as inclusions. Now, global sections are the elements of the kernel of the map

$$R[x^{\pm 1}, y] \times R[x, y^{\pm 1}] \to R[x^{\pm 1}, y^{\pm 1}]$$

that sends $(f, g) \mapsto f - g$. In other words

$$\mathcal{O}(U) = R[x^{\pm 1}, y] \cap R[x, y^{\pm 1}] = R[x, y].$$

If $U$ was affine, then the natural $U \to \mathrm{Spec}(R[x, y])$ would be an isomorphism, because it an inclusion of an open, an equality. But because $R \neq 0$, $\mathrm{Spec}(R[x, y] \setminus U) = \mathrm{Spec}(R)$ is non empty, a contradiction.

□